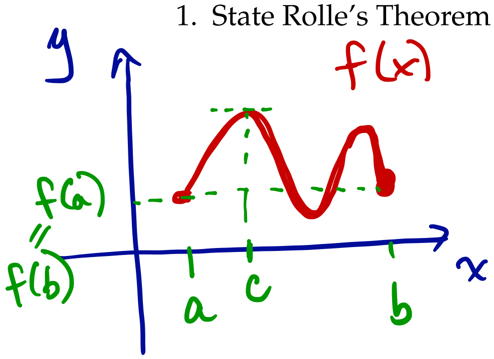
Lecture Notes: 4-2 The Mean Value Theorem (Part 2)

Warm-up Problems:

* You need all three hypotheses!

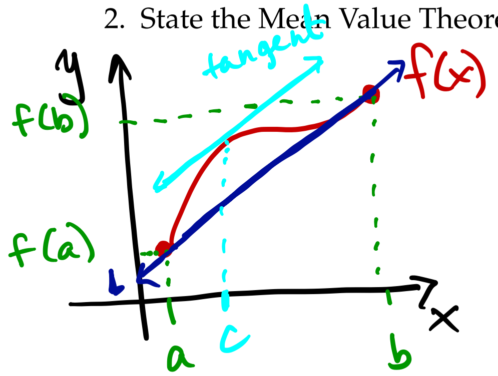
1. State Rolle's Theorem and draw a picture illustrating it.



If ① $f(a)=f(b)$ and ② $f(x)$ is continuous on $[a,b]$ and ③ $f(x)$ is differentiable on (a,b), then there exists some c in (a,b) where

$$f'(c)=0.$$

2. State the Mean Value Theorem and draw a picture illustrating it.



If $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b), then there exists some c in (a,b) so that

$$\underbrace{f'(c)}_{\text{Slope of tangent}} = \underbrace{\frac{f(b)-f(a)}{b-a}}_{\text{Slope of line } L}$$

3. Johnny Fever says "Rolle's Theorem? We don't need no stinking Rolle's Theorem. It's just a special case of the Mean Value Theorem." Is he right? Explain.

$\boxed{\text{Yes.}}$ Since Rolle's Thm has all the hypotheses of the Mean Value Theorem AND the additional hypothesis that $f(a)=f(b)$. So the right-hand side of the conclusion of MVP: $\frac{f(b)-f(a)}{b-a}=0$.

$\boxed{\text{No.}}$ We did USE Rolle's Theorem to show MVThm is true.

4. Consider $f(x) = 1/x$ on the interval $[1, 3]$.

(a) Verify that the function $f(x)$ satisfies the hypothesis of the Mean Value Theorem on the given interval.

$f(x)$ is continuous and differentiable on $(0,\infty)$. Thus it is continuous on $[1,3]$ and differentiable on $(1,3)$.

(b) Find all numbers c that satisfy the conclusion of the Mean Value Theorem.

Work

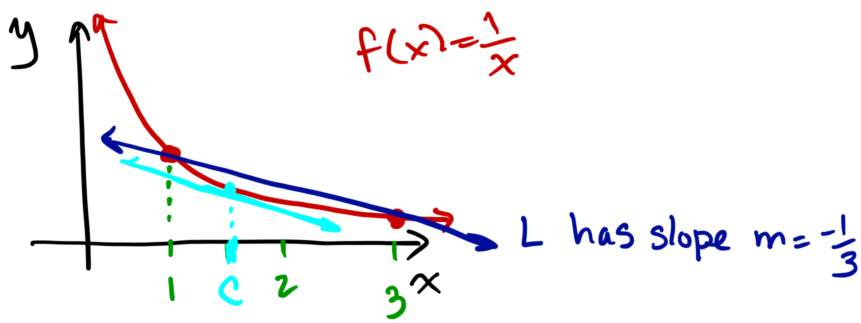
If $f(x)=\frac{1}{x}=x^{-1}$, then $f'(x)=-x^{-2}$. Since $a=1$, $f(a)=f(1)=1$.

Since $b=3$, $f(b)=f(3)=\frac{1}{3}$. So $\frac{f(b)-f(a)}{b-a}=\frac{\frac{1}{3}-1}{3-1}=\frac{-2/3}{2}=-\frac{1}{3}$.

We need c in $(1,3)$ so that $f'(c)=\frac{-1}{c^2}=\frac{-1}{3}$. So $c=\pm\sqrt{3}$. Ans: $c=+\sqrt{3}$.

$-\sqrt{3}$ not in $(1,3)$.

(c) Sketch the graph to show that your answer above are correct.

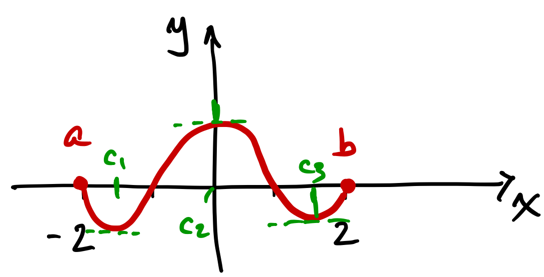


5. Construct an example of a specific function $f(x)$ and interval $[a, b]$ such that there are exactly three numbers c in (a, b) satisfying the Mean Value Theorem.

thinking:
I want exactly 3 "turn around" pts.
So construct a quartic.

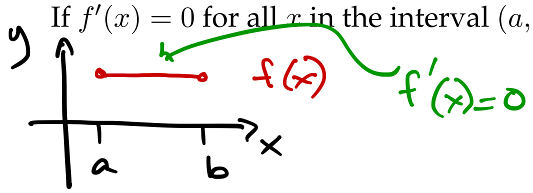
Ans: $f(x)=(x+2)(x+1)(x-1)(x-2)$ on $[-2,2]$

confirming picture:



6. Fill in the blank below and draw a picture illustrating this theorem.

If $f'(x) = 0$ for all x in the interval (a, b), then $f(x)=c$ on (a,b).



ONE LAST BIG IDEA:

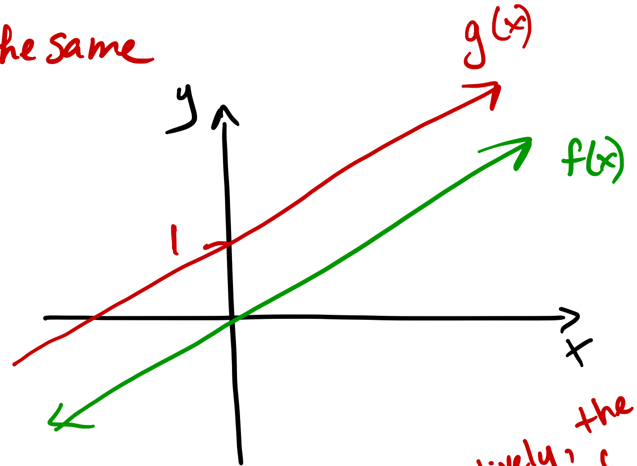
1. Give the formulas for two *different* functions $f(x)$ and $g(x)$ such that $f'(x) = g'(x)$ and sketch these two functions on the same set of axes.

$f(x) = x$ $\quad$ $f'(x) = 1$

$g(x) = x+1$ $\quad$ $g'(x) = 1$

not the same

the same



2. **Corollary 7:** If $f'(x) = g'(x)$ for all x in the interval (a, b), then

$$\boxed{f(x) = g(x) + c}$$

some constant

Intuitively, the graphs of f and g are translations of each other.

or $g(x) = f(x) + c$

or, said another way,

$$\boxed{f(x) - g(x) = c}$$

Intuitively f and g are a fixed distance apart

3. Why is Corollary 2 true?

Use #6 on previous page.

That is, if $\boxed{f'(x) = g'(x)}$, then $\boxed{f'(x) - g'(x) = 0.}$ ← Just algebra

on (a,b).

So $\boxed{\text{Thm } 5}$ or $\boxed{\#6}$ implies $f(x) - g(x) = c.$

or

$f(x) = g(x) + c.$

What really important property of the derivative are we using here?

PRACTICE PROBLEMS:

1. Suppose f is continuous on $[2,5]$ and $1 \le f'(x) \le 4$ for all x in $(2,5)$. Show that $3 \le f(5) - f(2) \le 12$.

thinking:

- $1 \le f'(x)$ means f must increase by at least $1 \cdot 3 = 3$ units on $[2,5]$
- $f'(x) \le 4$ means f can increase by at most $4 \cdot 3 = 12$ units on $[2,5]$
- $f(5) - f(2)$ is how much $f(x)$ increases on $[2,5]$

ANSWER: For all x's $\boxed{1 \le f'(x) \le 4.}$

The MVThm says there is some c so that

$$f'(c) = \frac{f(5)-f(2)}{5-2}$$

or, equivalently

$$(f'(c)) \cdot 3 = f(5) - f(2).$$

Using that first inequality in blue, we get

$$1 \cdot 3 \le f(5) - f(2) \le 4 \cdot 3,$$ or

$$3 \le f(5) - f(2) \le 12.$$

2. Suppose that $f(0) = -3$ and $f'(x) \le 5$ for all values of x. How large can $f(2)$ possibly be?

ANS Again, the MVThm applies since f is differentiable and therefore continuous on $(0,2)$. So there is some c in $(0,2)$ so that

$$f'(c) = \frac{f(2)-f(0)}{2-0} = \frac{f(2)-(-3)}{2}.$$

But $f'(c) \le 5$.

So $\dfrac{f(2)+3}{2} \le 5$. Thus, $f(2) \le -3+10 = 7$.

Thinking For each unit increase in x, y can increase by at most 5 (b/c $f'(x) \le 5$). So $f(x)$ can increase by at most $2 \cdot 5 = 10$.

3. For each function below, show that there is no value of c on $[0,2]$ such that $f'(c) = \dfrac{f(2)-f(0)}{2-0}$.
* Why does this not contradict Rolle's Theorem?

a) $f(x) = |x-1|$

$f(0)=1$, $f(2)=1$

$\dfrac{f(2)-f(0)}{2-0} = 0$

But $f'(x) = \begin{cases} 1 & \text{for } x>1 \\ -1 & \text{for } x<1 \end{cases}$

$1 \neq 0$ and $-1 \neq 0$

$f(x)$ is not differentiable on $(0,2)$. (f has no derivative at $x=1$.)

b) $f(x) = \dfrac{1}{(x-1)^2} = (x-1)^{-2}$

$f(0)=1$ and $f(2)=1$

So $\dfrac{f(2)-f(0)}{2-0} = 0$

But $f'(x) = -2(x-1)^{-3} = \dfrac{-2}{(x-1)^3}$.

Since $-2 \neq 0$, $f'(x) \neq 0$ for any x.

$f(x)$ is not continuous at $x=1$ in $[0,2]$

4. Two stationary patrol cars equipped with radar are 5 miles apart on a highway. As a truck passes the first patrol car, its speed is clocked at 55 miles per hour. Four minutes later, when the truck passes the second patrol car, its speed is clocked at 50 miles per hour. Prove that the truck must have exceeded the speed limit of 55 miles per hour at some time during the four minutes.

Assuming that the truck's speed and position are continuous and smooth functions of time (entirely reasonable assumptions :-)), the MV Thm applies.

(Get the units right: 4 minutes $= \frac{4}{60} = \frac{1}{15}$ hour)

So in $\frac{1}{15}$th of an hour, the truck travelled 5 miles. So its average velocity was: $\dfrac{\text{distance}}{\text{time}} = \dfrac{5\text{mi}}{\frac{1}{15}\text{hr}} = 75 \text{mi/hr}$. The MVThm says that there must be some time in $(0, \frac{1}{15})$ when $\underbrace{f'(c)}_{\text{velocity}} = 75\text{mi/hr}$.